On the Cohomology of the Moduli space of Parabolic Connections

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Introduction

In this poster, we study the moduli space of logarithmic connections of rank 2 on $\mathbb{P}^1 \setminus \{t_1, \ldots, t_5\}$ with fixed spectral data. We compute the cohomology of such moduli space, and this computation will be used to extend the results of geometric Langlands corespondence of [1] to the case where the parabolic connections have five simple poles on $\mathbb{P}^1$.

Preliminaries

We introduce $\mathfrak{sl}_2$-connections.
Fix complex numbers $\nu_1, \ldots, \nu_n \in \mathbb{C}$. Suppose that $\nu_1 \cdots \nu_n \neq 0$ and

$$\sum_{i=1}^{n} \epsilon_i \nu_i \notin \mathbb{Z} \tag{1}$$

for any $(\epsilon_i), \epsilon_i \in \{1, -1\}$.

Definition 2.1 A *$\boldsymbol{\nu}$-$\mathfrak{sl}_2$-parabolic connection on* $\mathbb{P}^1$ is a triplet (E, ∇, φ) such that
1. E is a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\nabla\colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$ is a connection, where $D := t_1 + \cdots + t_n$,
3. $\varphi\colon \bigwedge^2 E \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ is a horizontal isomorphism,
4. the residue $\mathsf{res}_{t_i}(\nabla)$ of the connection ∇ at t_i has eigenvalues $\nu_i^{\pm}$, $1 \leq i \leq n$.

Here, we put

$$\nu_i^{\pm} := \pm\nu_i \ \ (i = 1, \ldots, n-1), \ \nu_n^{+} := \nu_n, \ \nu_n^{-} := 1 - \nu_n.$$

Denote by $\mathcal{M}$ the moduli stack of $\boldsymbol{\nu}$-$\mathfrak{sl}_2$-parabolic connections on $\mathbb{P}^1$, and by M its coarse moduli space.

For such $\boldsymbol{\nu}$, the parabolic direction $l_i := \ker(\mathbf{res}_{t_i}(\nabla) - \nu_i) \subset E_{|t_i}$ is uniquely determined. So, we can get the forgetful map

$$\mathsf{Bun} : M \to P \ ; \ (E, \nabla, \varphi) \mapsto (E, \{l_i\})$$

where P is the coarse moduli space of undecomposable quasi-parabolic bundles $(E, \{l_i\})$ on $\mathbb{P}^1$.

Now, we consider the following stratification of M. By the irreducibility of $(E, \nabla, \varphi) \in M$, we have the following proposition.

Proposition 2.2 *For $(E, \nabla, \varphi) \in M$, we have*

$$E \simeq \mathcal{O}(k) \oplus \mathcal{O}(-k-1) \ \textit{where} \ 0 \leq k \leq \left[\frac{n-3}{2}\right].$$

Denote by M^k the subvariety of M where $E \simeq \mathcal{O}(k) \oplus \mathcal{O}(-k-1)$. Then

$$M = M^0 \cup \cdots \cup M^{[(n-3)/2]}.$$

Note that the stratum M^0 is a Zariski open dense of M.

Geometric description of M^0

Suppose $n = 5$. For computation of the cohomology of M, we introduce some blowing-up of the Hirzebruch surface $\mathbb{F}_3$. Put $L := \Omega^1_{\mathbb{P}^1}(D)$. Let $\mathbb{L}$ be the total space of the line bundle L. Note that $\mathbb{L} = \mathbb{F}_3 \setminus s_\infty$ where s_∞ is the infinity section $(s_\infty)^2 = -3$.
Let $\pi\colon \mathbb{L} \to \mathbb{P}^1$ be the projection and let $\tau_i\colon F_i := \pi^{-1}(t_i) \xrightarrow{\sim} \mathbb{C}$ be the residue map. Put $\nu_i^{+} := \nu_i$, $\nu_i^{-} := -\nu_i$ for $i = 1, \ldots, 4$, $\nu_5^{+} := \nu_5, \nu_5^{-} := 1 - \nu_5$, and $\hat{\nu}_i^{\pm} := \tau_i^{-1}(\nu_i^{\pm})$.
Set

$$\mathcal{K}_5' := \left(\mathrm{Bl}_{\hat{\nu}_i^{\pm}}\mathbb{L}\right) \setminus (\widetilde{F}_1 \cup \cdots \cup \widetilde{F}_5)$$

where $\mathrm{Bl}_{\hat{\nu}_i^{\pm}}\mathbb{L}$ is the blowing-up of $\mathbb{L}$ at $\hat{\nu}_i^{\pm}$ for $i = 1, \ldots, n$, and $\widetilde{F}_i$ are the proper pre-images of the fiber F_i $i = 1, \ldots, n$.

Lemma 3.1

$$H^i(\mathcal{K}_5', \mathcal{O}_{\mathcal{K}_5'}) = \begin{cases} \mathbb{C}, & i = 0, \\ H^2_m(A) \neq 0, & i = 1, \\ 0, & i \geq 2, \end{cases}$$

where $(A, \mathfrak{m})$ is a local ring such that $\dim(A_{\mathfrak{m}}) = 2$.

We denote by $\mathcal{K}_5$ the image of $\mathcal{K}_5'$ under the projection $\mathcal{K}_5' \to \mathbb{F}_3 \setminus s_\infty$. We can define the following map

$$\begin{aligned} M^0 &\longrightarrow \mathrm{Sym}^2(\mathcal{K}_5) \\ (E, \nabla, \varphi) &\longmapsto \{(q_1, p_1), (q_2, p_2)\}, \end{aligned} \tag{2}$$

We consider the composite of the Hilbert-Chow morphism and the blowing-up

$$\mathrm{Hilb}^2(\mathcal{K}_5') \longrightarrow \mathrm{Sym}^2(\mathcal{K}_5') \longrightarrow \mathrm{Sym}^2(\mathcal{K}_5),$$

We have the next proposition.

Proposition 3.2 ([3] Theorem 5.2) *We can extend the map (2) to*

$$M^0 \longrightarrow \mathrm{Hilb}^2(\mathcal{K}_5')$$

and this map is injective.

Cohomology of $\mathcal{K}'_{5,q_1}$

We denote by $Z \subset \mathrm{Sym}^2(\mathcal{K}_5')$ the proper pre-image of $\{(q_1, p_1), (q_1, -p_1)\} \subset \mathrm{Sym}^2(\mathcal{K}_5')$ under the blowing-up $\mathrm{Sym}^2(\mathcal{K}_5') \to \mathrm{Sym}^2(\mathcal{K}_5)$, and by $\widetilde{Z} \subset \mathrm{Hilb}^2(\mathcal{K}_5')$ the proper pre-image of Z under the Hilbert-Chow morphism $\mathrm{Hilb}^2(\mathcal{K}_5') \to \mathrm{Sym}^2(\mathcal{K}_5')$. Denote by

$$\widetilde{\mathrm{Hilb}}^2(\mathcal{K}_5') \to \mathrm{Hilb}^2(\mathcal{K}_5') \tag{3}$$

the blowing-up along $\widetilde{Z}$ and by $\widehat{Z}$ the strict transform of $\widetilde{Z}$. We also denote by $(\mathcal{K}_5' \times \mathcal{K}_5')^\sim$ the blowing-up of $\mathcal{K}_5' \times \mathcal{K}_5'$ along the ideal $(q_1 - q_2, p_1 - p_2)$, and by $(\mathcal{K}_5' \times \mathcal{K}_5')^\approx$ the blowing-up of $(\mathcal{K}_5' \times \mathcal{K}_5')^\sim$ along the ideal $(q_1 - q_2, p_1 + p_2)$. Then $\mathrm{Hilb}^2(\mathcal{K}_5') = (\mathcal{K}_5' \times \mathcal{K}_5')^\sim / \mathfrak{S}_2$, and $\widetilde{\mathrm{Hilb}}^2(\mathcal{K}_5') = (\mathcal{K}_5' \times \mathcal{K}_5')^\approx / \mathfrak{S}_2$.
Now, using above description, we define another important blowing-up of Hirzebruch surface $\mathbb{F}_3$. Fix $q_1 \in \mathbb{P}^1 \setminus \{t_1, \cdots, t_5\}$ and define the fiber F_6 over q_1. We denote by $(\mathbb{F}_3)^\approx$ the blowing-up of $\widetilde{\mathbb{F}_3}$ at two points $\{(q_1, p_1), (q_1, -p_1)\}$ (when $p_1 = p_2 = 0$, blow up twice at $(q_1, 0)$). Set

$$\mathcal{K}'_{5,q_1} := (\mathbb{F}_3)^\approx \setminus (\widetilde{s}_\infty \cup \widetilde{F}_1 \cup \cdots \cup \widetilde{F}_6)$$

where $\widetilde{F}_6$ is the strict transform of F_6. We denote by $\mathcal{K}_{5,q_1}$ the image of $\mathcal{K}'_{5,q_1}$ under the projection $\mathcal{K}'_{5,q_1} \to \mathbb{F}_3 \setminus s_\infty$.
Define

$$D_{q_1} := 2s_\infty + \widetilde{F}_1 + \cdots + \widetilde{F}_5 + \widetilde{F}_6$$

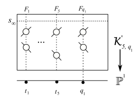
Then, we have

$$(s_\infty, \widetilde{F}_i) = 1, (\widetilde{F}_i, \widetilde{F}_j) = 0, (s_\infty, s_\infty) = -3, (\widetilde{F}_i, \widetilde{F}_i) = -2$$

This implies

$$(D_{q_1}, D_{q_1}) = (D_{q_1}, s_\infty) = (D_{q_1}, \widetilde{F}_i) = 0.$$

This is the same condition of Arinkin Lysenko [2], section 5.2.



Proposition 4.1

$$H^i(\mathcal{K}'_{5,q}, \mathcal{O}_{\mathcal{K}'_{5,q}}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Cohomology of M

Set $\widehat{M}_Z := M^0 \cup \widehat{Z}$. By Proposition 3.2, we have injective maps $\iota : M^0 \hookrightarrow \mathrm{Hilb}^2(\mathcal{K}_5')$ and $\hat{\iota} : \widehat{M}_Z \hookrightarrow \widetilde{\mathrm{Hilb}}^2(\mathcal{K}_5')$. We define the blowing-up parameter λ_- as $p_1 + p_2 = \lambda_-(q_1 - q_2)$.
Set $T := \widetilde{\mathrm{Hilb}}^2(\mathcal{K}_5') \setminus \widehat{M}_Z$. For a vector bundle $\mathcal{F}$ on $\widetilde{\mathrm{Hilb}}^2(\mathcal{K}_5')$,

$$\begin{aligned} H^i(\widehat{M}_Z, \mathcal{F}|_{\widehat{M}_Z}) &= H^i(\widetilde{\mathrm{Hilb}}^2(\mathcal{K}_5'), \hat{\iota}_*\hat{\iota}^*\mathcal{F}) \\ &= \varinjlim H^i(\widetilde{\mathrm{Hilb}}^2(\mathcal{K}_5'), \mathcal{F}(kT)). \end{aligned}$$

To compute $H^i(\widetilde{\mathrm{Hilb}}^2(\mathcal{K}_5'), \mathcal{F}(kT))$, consider $H^i((\mathcal{K}_5' \times \mathcal{K}_5')^\approx, \mathcal{F}(kT'))$, where T' is defined by $(\lambda_- = \infty)$. We can define a map

$$\begin{aligned} f : (\mathcal{K}_5' \times \mathcal{K}_5')^\approx \setminus T' &\longrightarrow \mathcal{K}_5' \\ (q_1, p_1, q_2, p_2) &\longmapsto (q_1, p_1), \end{aligned}$$

and the fiber $f^{-1}(\{(q_1, p_1)\}) \simeq \mathcal{K}'_{5,q_1}$. By Leray's spectral sequence, we have

$$H^i((\mathcal{K}_5' \times \mathcal{K}_5')^\approx \setminus T', \mathcal{F}) \simeq \bigoplus_{p+q=i} H^p(\mathcal{K}_5', R^q f_*\mathcal{F}).$$

Using the Base change theorem, we have $(R^q f_*\mathcal{F})_{(q_1,p_1)} \simeq H^q(\mathcal{K}'_{5,q_1}, \mathcal{F}_{(q_1,p_1)})$.
By using Lemma 3.1 and Proposition 4.1, we have

$$H^i((\mathcal{K}_5' \times \mathcal{K}_5')^\approx \setminus T', \mathcal{O}) = \begin{cases} \mathbb{C}, & i = 0, \\ H^2_{\mathfrak{m}}(A) \neq 0, & i = 1, \\ 0, & i > 1. \end{cases}$$

Moreover, the action of $\mathfrak{S}_2$ on $H^1((\mathcal{K}_5' \times \mathcal{K}_5')^\approx \setminus T', \mathcal{O})$ is nontrivial. Therefore,

$$H^i(\widehat{M}_Z, \mathcal{O}_{\widehat{M}_Z}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Since $\mathrm{codim}_{\mathrm{Hilb}^2(\mathcal{K}_5')}(\widetilde{Z}) = 2$, and $M^1 = M \setminus M^0 \simeq \mathbb{A}^2$, we have

Theorem 5.1

$$H^i(M, \mathcal{O}_M) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

References

[1] D. Arinkin, *Orthogonality of natural sheaves on moduli stacks of SL(2)-bundles with connections on P1 minus 4 points.*, Selecta Math., New Series 7 (2001), 213-239.

[2] D. Arinkin, S. Lysenko, *On the moduli of $SL(2)$-bundles with connections on* $\mathbb{P}^1 \setminus \{x_1, \ldots, x_4\}$, Internat. Math. Res. Notices (1997), no. **19**, 983–999.

[3] A. Komyo, M.-H. Saito, *Explicit description of jumping phenomena on moduli spaces of parabolic connections and Hilbert schemes of points on surfaces, accepted in Kyoto Journal of Mathematics.* (arXiv:math/1611.00971)